Quantum anomaly and geometric phase; their basic differences

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Abstract

It is sometimes stated in the literature that the quantum anomaly is regarded as an example of the geometric phase. Though there is some superficial similarity between these two notions, we here show that the differences bewteen these two notions are more profound and fundamental. As an explicit example, we analyze in detail a quantum mechanical model proposed by M. Stone, which is supposed to show the above connection. We show that the geometric term in the model, which is topologically trivial for any finite time interval T, corresponds to the so-called "normal naive term" in field theory and has nothing to do with the anomaly-induced Wess-Zumino term. In the fundamental level, the difference between the two notions is stated as follows: The topology of gauge fields leads to level crossing in the fermionic sector in the case of chiral anomaly and the *failure* of the adiabatic approximation is essential in the analysis, whereas the (potential) level crossing in the matter sector leads to the topology of the Berry phase only when the precise adiabatic approximation holds.

1 Introduction

In quantum field theory the quantum anomaly plays an important role to test if a specific classical symmetry in question is really preserved in quantum theory [1, 2, 3, 4]. The quantum anomaly also predicts some novel phenomena which are not expected by a classical consideration, for example, the baryon number violation in the Weinberg-Salam theory [5]. In some special cases of chiral anomaly, one can summarize the effects of the quantum anomaly in the form of an extra Wess-Zumino term [6] which is added to the starting Lagrangian.

On the other hand, it has been recognized that one obtains phase factors in the adiabatic treatment (such as in the Born-Oppenheimer approximation) of the Schrödinger equation which depends on slowly varying background variables [7] -[17]. These phases are called "geometric phases", and they are generally associated with level crossing. Although the manner of obtaining geometric phases is quite different from that of quantum anomalies,

it is sometimes stated in the literature that the chiral anomaly is regarded as a kind of geometric phase [18, 19].

The notion of the geometric phase itself does not appear to be sharply defined at this moment. In the influential book by Shapere and Wilczek [20], various phase factors in physics which exhibit topological properties are discussed together as geometric phases. It is important to synthesize various phenomena and notions into a unifying notion, but it is the opinion of the present author that this broad use of the scientific terminology could lead to confusions and mis-understandings in view of the wide use of geometric phases in various fields in physics today. This broad use of the terminology is closely related to the broad use of the terminology of "adiabatic approximation". The practical Born-Oppenheimer approximation, which provides a typical adiabatic approximation in physics, contains two quite different time scales but the slower time scale T measured in units of the time scale of the faster system is *finite*. In such a practical Born-Oppenheimer approximation, it is shown that the commonly referred Berry's phase, which is purely dynamical without any approximation, is topologically trivial and no monopole-like singularity at the level crossing point [21]. The notion such as holonomy is valid for the level crossing problem only in the precise adiabatic limit with $T \to \infty$ [11].

The above properties of the geometric phase become quite clear in a recent attempt to formulate the geometric phase in the second quantized formulation [21]. This approach works in both of path integral and operator formulations, and the analysis of geometric phases is reduced to a simple diagonalization of the Hamiltonian. The hidden local gauge symmetry, which arises from the fact that the choice of basis vectors in the functional space is arbitrary in field theory, replaces the notions of parallel transport and holonomy [22]. By carefully diagonalizing the geometric term in the infinitesimal neighborhood of level crossing, it is shown that the topological property of the geometric phase is trivial in the practical Born-Oppenheimer approximation, where the period T of the slower system is finite, and thus no monopole-like singularity, as already stated above. This approximate topology in the geometric phase is quite different from the exact topology associated with gauge fields such as in the familiar Aharonov-Bohm effect [23]. We thus become somewhat suspicious about the claim on the equivalence of quite distinct notions such as quantum anomaly and geometric phase. The purpose of the present paper is to show that these two notions, namely, quantum anomaly and geometric phase, may have some superficial similarity to each other, but the differences in these two notions are more profound and fundamental.

In the literature, the paper by M. Stone [18] is often quoted as an evidence of the equivalence of the quantum anomaly and the geometric phase. We thus explain the crucial differences between the geometric phase and

the quantum anomaly by taking the model by Stone [1] as a concrete example, though our analysis is valid for the more general model summarized in Appendix. We first analyze the problem from the point of view of several characteristic properties of the chiral anomaly [2], such as the failure of the naive manipulation and the failure of the decoupling theorem, on the basis of the explicit model in [18] and a corresponding field theoretical model which contains a true anomaly in Sections 2 and 3. We show that the interpretation of the geometric term in the model in [18] as the Wess-Zumino term, namely, a manifestation of quantum anomaly is untenable even in the precise adiabatic approximation. We then analyze the problem from the point of view of two key concepts involved in both of the chiral anomaly and the geometric phase, namely, level crossing and topology. By a careful examination of the statements made in the paper by Nelson and Alvarez-Gaume [24], we explain in Section 4 that the chiral anomaly and the geometric phase are completely different in the fundamental level.

2 Quantum mechanical model; geometric phase

We first recapitulate the model due to M. Stone[18]. The model starts with the Hamiltonian

$$H = \frac{\vec{L}^2}{2I} - \mu \mathbf{n}(t) \cdot \vec{\sigma} \tag{2.1}$$

where $\mathbf{n}(t)$ is a unit vector specifying the direction of the "magnetic field" acting on the spin represented by the Pauli matrix $\vec{\sigma}$, and $\vec{L}$ generates the rotation of $\mathbf{n}(t)$. We analyze the mathematical aspects of the model (2.1) in this paper without asking the possible physical meaning of the specific model, which is explained in [18]. Partly referring to the second quantization, one can write the above Hamiltonian as

$$H(t) = \frac{\vec{L}^2}{2I} - \psi^\dagger \mu \mathbf{n}(t) \cdot \vec{\sigma} \psi \tag{2.2}$$

where the field ψ stands for the two-component spinor.

One may then write an evolution operator in the formal path integral representation [18]

$$\langle f| \exp[-\frac{i}{\hbar}\int_0^T H dt]|i\rangle$$

[1] It should however be emphasized that we are not criticizing the analysis of the geometric phase itself in the model by Stone, which is essentially identical to the simplest example in [10].

[2] To make our analysis definite, we define the quantum anomaly as the even-dimensional chiral anomaly and the geometric phase as the phase associated with general level crossing which is summarized in Appendix.

$$
\begin{aligned}
&= \int \mathcal{D}\vec{n}\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\delta(\vec{n}^2-1) \\
&\times \exp\{\frac{i}{\hbar}\int_0^T dt[\frac{\dot{\vec{n}}^2}{2I}+\psi^{\dagger}\frac{\hbar}{i}\partial_t\psi+\psi^{\dagger}\mu\vec{n}(t)\cdot\vec{\sigma}\psi]\}
\end{aligned}
\tag{2.3}
$$

or in the Euclidean formulation ($t \to -i\tau$), we have

$$
\begin{aligned}
&\int \mathcal{D}\vec{n}\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\delta(\vec{n}^2-1) \\
&\times \exp\{\frac{1}{\hbar}\int_0^{\beta} d\tau[-\frac{\dot{\vec{n}}^2}{2I}+\psi^{\dagger}\hbar\partial_{\tau}\psi+\psi^{\dagger}\mu\vec{n}(\tau)\cdot\vec{\sigma}\psi]\}.
\end{aligned}
\tag{2.4}
$$

Following Ref. [18], we take this path integral as our starting point.

This path integral is rewritten as

$$
\begin{aligned}
&\int \mathcal{D}\vec{n}\mathcal{D}\psi'^{\dagger}\mathcal{D}\psi'\delta(\vec{n}^2-1) \\
&\times \exp\{\frac{1}{\hbar}\int_0^{\beta} d\tau[-\frac{\dot{\vec{n}}^2}{2I}+\psi'^{\dagger}(\hbar\partial_{\tau}+\mu\sigma_3+U(\vec{n}(\tau))^{\dagger}\hbar\partial_{\tau}U(\vec{n}(\tau)))\psi']\} \\
&= \int \mathcal{D}\vec{n}\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\delta(\vec{n}^2-1) \\
&\times \exp\{\frac{1}{\hbar}\int_0^{\beta} d\tau[-\frac{\dot{\vec{n}}^2}{2I}+\psi^{\dagger}(\hbar\partial_{\tau}+\mu\sigma_3+U(\vec{n}(\tau))^{\dagger}\hbar\partial_{\tau}U(\vec{n}(\tau)))\psi]\}
\end{aligned}
\tag{2.5}
$$

when one performs a time-dependent unitary transformation (or a gauge transformation)

$$
\begin{aligned}
\psi(\tau) &= U(\vec{n}(\tau))\psi'(\tau), \\
\psi^{\dagger}(\tau) &= \psi'^{\dagger}(\tau)U^{\dagger}(\vec{n}(\tau))
\end{aligned}
\tag{2.6}
$$

with

$$
U(\vec{n}(\tau))^{\dagger}\vec{n}(\tau)\vec{\sigma}U(\vec{n}(\tau)) = |\vec{n}|\sigma_3. \tag{2.7}
$$

The last relation in (2.5) means that the naming of integration variables is arbitrary in the path integral. An explicit form of the unitary transformation is given by defining

$$
\begin{aligned}
v_{+}(\vec{n}) &= \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}, \\
v_{-}(\vec{n}) &= \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix}
\end{aligned}
\tag{2.8}
$$

in terms of the polar coordinates, $n_1 = |\vec{n}|\sin\theta\cos\varphi$, $n_2 = |\vec{n}|\sin\theta\sin\varphi$, $n_3 = |\vec{n}|\cos\theta$. Note that these eigenfunctions, which satisfy

$$\mu\vec{n}(\tau)\vec{\sigma}v_{\pm}(\vec{n}) = \pm\mu|\vec{n}|v_{\pm}(\vec{n}), \tag{2.9}$$

are singular at the origin $\mu\vec{n} = 0$ and also contain spurious singularities at north and south poles [3]. In our choice of the phase convention, we have $v_{\pm}(\vec{n}(0)) = v_{\pm}(\vec{n}(\beta))$ if $\vec{n}(0) = \vec{n}(\beta)$; it has been explained in detail elsewhere [22] and later in the appendix that the choice of the time-dependent phases of these eigenfunctions is arbitrary due to the hidden local gauge symmetry. Then $U(\vec{n}(\tau))$ is given by a 2×2 matrix

$$U(\vec{n}(\tau)) = \begin{pmatrix} v_+(\vec{n}) & v_-(\vec{n}) \end{pmatrix}. \tag{2.10}$$

This unitary transformation keeps the path integral measure invariant

$$\mathcal{D}\psi^{\dagger}\mathcal{D}\psi = \mathcal{D}\psi'^{\dagger}\mathcal{D}\psi' \tag{2.11}$$

without giving a non-trivial Jacobian for the present two-component problem (2.6), as long as $U(\vec{n}(\tau))$ is not singular. The matrix $U(\vec{n}(\tau))$ becomes singular at the level crossing point which takes place at $\mu\vec{n} = 0$ in the present case. (In terms of the polar coordinates, $U(\vec{n}(\tau))$ at the north or south pole exhibits spurious singularity.) The treatment in the infinitesimal neighborhood of the singularity is discussed later.

If one defines (in Euclidean metric, but the result is valid for Minkowski metric also)

$$v_m^{\dagger}(\vec{n})i\frac{\partial}{\partial\tau}v_n(\vec{n}) = A^k_{mn}(\vec{n})\dot{n}_k \tag{2.12}$$

where m and n run over $\pm$, we have

$$\begin{aligned}
A^k_{++}(\vec{n})\dot{n}_k &= \frac{(1+\cos\theta)}{2}\dot{\varphi},\\
A^k_{+-}(\vec{n})\dot{n}_k &= \frac{\sin\theta}{2}\dot{\varphi} + \frac{i}{2}\dot{\theta} = (A^k_{-+}(\vec{n})\dot{n}_k)^{\star},\\
A^k_{--}(\vec{n})\dot{n}_k &= \frac{1-\cos\theta}{2}\dot{\varphi}.
\end{aligned} \tag{2.13}$$

Note that we have

$$\mathrm{Tr}[v_m^{\dagger}(\vec{n})i\frac{\partial}{\partial\tau}v_n(\vec{n})] = \dot{\varphi}. \tag{2.14}$$

[3] In the context of level crossing, it is natural to consider the combination $\mu\vec{n}$ by allowing the possible time dependence of $\mu(t)$. If the variable $\mu(t)\vec{n}$ moves toward the origin during a cyclic motion, it implies that the two levels appraoch the level crossing point.

The above relation (2.5) implies the equivalence of two Lagrangians

$$L = -\frac{\dot{\vec{n}}^2}{2I} + \psi^\dagger \hbar \partial_\tau \psi + \psi^\dagger \mu \vec{n}(\tau) \cdot \vec{\sigma} \psi \tag{2.15}$$

and

$$L' = -\frac{\dot{\vec{n}}^2}{2I} + \psi^\dagger (\hbar \partial_\tau + \mu \sigma_3 + U(\vec{n}(\tau))^\dagger \hbar \partial_\tau U(\vec{n}(\tau)))\psi. \tag{2.16}$$

The fermionic part of the starting Hamiltonian (2.2) is thus equivalent to [4] (by going back to Minkowski metric)

$$H_{\text{fermion}}(t) = -\psi^\dagger [\mu \sigma_3 + U(\vec{n}(t))^\dagger \frac{\hbar}{i} \partial_t U(\vec{n}(t))|_{|\vec{n}|=1}]\psi \tag{2.17}$$

or in the original notation of (2.1)

$$H_{\text{fermion}} = -\mu \sigma_3 - U(\vec{n}(t))^\dagger \frac{\hbar}{i} \partial_t U(\vec{n}(t))|_{|\vec{n}|=1}. \tag{2.18}$$

The last term in (2.18), which may be understood as a pure gauge term, is generally called as "geometric term" for the historical reason. The survival of this geometric term in the limit of the large μ limit was interpreted in Ref. [18] as an evidence of the failure of the decoupling theorem. The failure of the decoupling theorem in the context of quantum anomaly is however more involved, as will be explained later. This Hamiltonian (2.18), which is exact, carries all the information about the geometric phases as we show below; this means that the geometric phases are purely *dynamical*.

If one is interested in the lower energy state of the Hamiltonian (2.18), one has an approximate Hamiltonian

$$\begin{aligned} H_{ad} &\simeq -\mu - (U(\vec{n}(t))^\dagger \frac{\hbar}{i} \partial_t U(\vec{n}(t))|_{|\vec{n}|=1})_{++} \\ &= -\mu + \hbar \frac{(1+\cos\theta)}{2} \dot{\varphi} \end{aligned} \tag{2.19}$$

by noting (2.13). If μ is sufficiently large, to be precise for

$$2\mu T \gg 2\pi\hbar, \tag{2.20}$$

[4]The Legendre transformation from the Lagrangian to the total Hamiltonian is involved in the presence of the derivative coupling as in the present example (2.16). Thus our fermionic Hamiltonian (2.17) is valid only when the variable $\vec{L}$ or $n(t)$ is treated as a background c-number. This limitation, however, does not influence our analysis of the possible connection of the geometric term with the Wess-Zumino term. The analysis of geometric term is generally performed in this simplified situation [17]. The second quantized path integral approach to the geometric term [21] is more flexible for the treatment of more general situations.

one may neglect the off-diagonal part in the geometric term in (2.18), and this Hamiltonian H_{ad} provides a good adiabatic approximation to the full Hamiltonian. Here T is the period of the slower dynamical system $\vec{n}(t)$ and $2\pi\hbar$ stands for the magnitude of the geometric term times T. We emphasize that the adiabatic approximation in the present context corresponds to throwing away the off-diagonal part in the geometric term, namely, throwing away *a part of the Hamiltonian*. The geometric term in (2.19) is reminiscent of a magnetic monopole located in the parameter space at the level crossing point $\mu\vec{n} = 0$. The fermionic Hamiltonian (2.19) thus gives rise to the dynamical phase

$$\begin{aligned}
&\exp\{-\frac{i}{\hbar}\int_0^T dt[-\mu + \hbar\frac{(1+\cos\theta)}{2}\dot{\varphi}]\} \\
&= \exp\{\frac{i}{\hbar}\mu T - i\oint\frac{(1+\cos\theta)}{2}d\varphi]\}
\end{aligned} \tag{2.21}$$

for a cyclic motion of the slower sytem, and the second term gives rise to the familiar Berry's phase [18, 10].

The last geometric term in (2.18) has an *approximate* topological property around the level crossing point in the practical Born-Oppenheimer approximation where the period of the slower dynamical system T is finite. This fact is understood as follows: For sufficiently close to the level crossing point, $\mu \sim 0$ but $\mu \neq 0$, one has $\mu T \ll 2\pi\hbar$ instead of (2.20). One may then perform a further unitary transformation of the fermionic variable [21]

$$\begin{aligned}
\psi'(t) &= U(\theta(t))\psi''(t), \\
\psi'(t)^\dagger &= \psi''^\dagger(t)U^\dagger(\theta(t))
\end{aligned} \tag{2.22}$$

with

$$U(\theta(t)) = \left(\begin{array}{cc} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{array}\right) \tag{2.23}$$

in addition to (2.6). The Hamiltonian (2.17) is thus equivalent to (by repeating the path integral analysis)

$$\begin{aligned}
H_{\text{fermion}}(t) &= -\psi^\dagger[\mu U(\theta(t))^\dagger\sigma_3 U(\theta(t)) \\
&\quad +(U(\theta(t))U(\vec{n}(t)))^\dagger\frac{\hbar}{i}\partial_t(U(\vec{n}(t))U(\theta(t)))|_{|\vec{n}|=1}]\psi \\
&= -\psi^\dagger[\mu U(\theta(t))^\dagger\sigma_3 U(\theta(t)) - \hbar\left(\begin{array}{cc} \dot{\varphi} & 0 \\ 0 & 0 \end{array}\right)]\psi \\
&\simeq +\psi^\dagger\hbar\left(\begin{array}{cc} \dot{\varphi} & 0 \\ 0 & 0 \end{array}\right)\psi
\end{aligned} \tag{2.24}$$

for $\mu \sim 0$, or in the original notation

$$H_{\rm fermion} \simeq \hbar \begin{pmatrix} \dot{\varphi} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.25}$$

The geometric phase thus either vanishes or becomes trivial

$$\exp\{-i\int_0^T \dot{\varphi} dt\} = \exp\{-2i\pi\} = 1 \tag{2.26}$$

in the infinitesimal neighborhood of level crossing. The geometric term is thus topologically (i.e., under the continuous variation of the parameter μ) trivial for any finite T. At the level crossing point, $\mu \sim 0$, the conventional energy becomes degenerate but the degeneracy is lifted when one diagonalizes the geometric term. It is important that the additional transformation (2.23) depends on the variable θ only and preserves (2.14).

Though the geometric phase is topologically trivial in a precise sense, it is still interesting that the geometric phase is approximately topological. This approximate topological property (of a pure gauge term) is traced to the fact that the eigenfunctions in (2.9) are singular on top of the level crossing, *i.e.*, the gauge transformation (2.6) is singular, though the singular behavior is avoided in the sense of the trivial phase as in (2.25) by defining a suitable basis set in the neighborhood of the singularity by a further unitary transformation. (The above relation (2.24) also shows that if the time variation of $\vec{n}(t)$ is faster than the fermionic variables even for μ which is not small, the geometric term dominates the $\mu\sigma_3$ term, and the geometric term becomes topologically trivial. This is another idication that the geometric term is not quite topological, and this observation becomes relevant when one compares the geometric phase with the chiral anomaly.)

The geometric term corresponds to the *normal term* not an anomalous term in field theory, as we explain in Section 3. The geometric term in the present model has nothing to do with the Wess-Zumino term as we understand it in field theory which is a result of the symmetry breaking by quantum effects. To be more precise, (2.5) shows that

$$\begin{aligned} &\det[\hbar\partial_\tau + \mu\vec{n}(\tau)\cdot\vec{\sigma}] \\ &= \det[U(\vec{n}(\tau))^\dagger\hbar\partial_\tau + \mu\vec{n}(\tau)\cdot\vec{\sigma}\}U(\vec{n}(\tau))] \\ &= \det[\hbar\partial_\tau + \mu\sigma_3 + U(\vec{n}(\tau))^\dagger\hbar\partial_\tau U(\vec{n}(\tau))]. \end{aligned} \tag{2.27}$$

The ordinary Wess-Zumino term would manifest itself as an *extra* phase factor on the right-hand side of this relation (see, for example, (3.25)), but no such an extra phase in the present example.

This analysis of the Wess-Zumino term becomes more transparent if one considers $H = \frac{\vec{L}^2}{2I} - \mu\mathbf{n}(t)\cdot\vec{\sigma} + \mu_0$ instead of (2.1), or

$$H = \frac{\vec{L}^2}{2I} - \psi^\dagger[\mu\mathbf{n}(t)\cdot\vec{\sigma} - \mu_0]\psi \tag{2.28}$$

instead of (2.2) with a positive constant μ_0 which satisfies

$$\mu_0 > \mu \tag{2.29}$$

by noting that the absolute value of the energy is not fixed in the present quantum mechanical model. This choice incidentally defines a Euclidean theory more precisely, as the Hamiltonian becomes positive definite. Then the equivalent Hamiltonian (by treating $\vec{n}(t)$ as a background variable) in (2.17) is replaced by

$$H_{\text{fermion}}(t) = -\psi^\dagger[\mu\sigma_3 - \mu_0 + U(\vec{n}(t))^\dagger \frac{\hbar}{i}\partial_t U(\vec{n}(t))|_{|\vec{n}|=1}]\psi \tag{2.30}$$

or in the original notation of (2.1)

$$H_{\text{fermion}} = -\mu\sigma_3 + \mu_0 - U(\vec{n}(t))^\dagger \frac{\hbar}{i}\partial_t U(\vec{n}(t))|_{|\vec{n}|=1}. \tag{2.31}$$

The adiabatic approximation for the lower energy state $|+\rangle$ is then given by

$$\begin{aligned} H_{ad} &\simeq -\mu + \mu_0 - (U(\vec{n}(t))^\dagger \frac{\hbar}{i}\partial_t U(\vec{n}(t))|_{|\vec{n}|=1})_{++} \\ &= -\mu + \mu_0 + \hbar\frac{(1+\cos\theta)}{2}\dot{\varphi} \end{aligned} \tag{2.32}$$

and the dynamical phase for the fermionic part is given by

$$\begin{aligned} &\exp\{-\frac{i}{\hbar}\int_0^T dt[-\mu + \mu_0 + \hbar\frac{(1+\cos\theta)}{2}\dot{\varphi}]\} \\ &= \exp\{-\frac{i}{\hbar}(\mu_0 - \mu)T - i\oint \frac{(1+\cos\theta)}{2}d\varphi]\}. \end{aligned} \tag{2.33}$$

We thus obtain the same geometric phase independently of μ_0. The almost topological property of the geometric phase arises from the crossing of two levels

$$\mu_0 \pm \mu|\vec{n}(t)| > 0 \tag{2.34}$$

at $\mu|\vec{n}(t)| = 0$; the crossing of *positive* and *negative* levels at $\mu|\vec{n}(t)| = 0$, which is realized when one sets $\mu_0 = 0$, is not essential for the geometric phase. The fact that we can include an arbitrary mass parameter μ_0 shows

that the basic symmetry in the present model is *vector-like* which contains no anomaly, to be consistent with the absence of the non-trivial Jacobian. This may be compared to (3.7).

For the present case (2.28) also, we have a naive relation

$$\begin{aligned}
&\det[\hbar\partial_\tau + \mu\vec{n}(\tau)\cdot\vec{\sigma} - \mu_0] \\
&= \det[U(\vec{n}(\tau))^\dagger\{\hbar\partial_\tau + \mu\vec{n}(\tau)\cdot\vec{\sigma} - \mu_0\}U(\vec{n}(\tau))] \\
&= \det[\hbar\partial_\tau + \mu\sigma_3 - \mu_0 + U(\vec{n}(\tau))^\dagger\hbar\partial_\tau U(\vec{n}(\tau))]
\end{aligned} \tag{2.35}$$

without any extra phase factor which would correspond to the Wess-Zumino term. We also have for H_{fermion} in (2.30)

$$\langle 0|\exp\{-\frac{1}{\hbar}\int_0^\beta H_{\text{fermion}}(\tau)d\tau\}|0\rangle = 1 \tag{2.36}$$

for the fermionic vacuum $|0\rangle$ in the second quantized sense defined by

$$\psi_+|0\rangle = \psi_-|0\rangle = 0 \tag{2.37}$$

in the adiabatic picture where one can approximately diagonalize the fermionic Hamiltonian by treating the variable $\vec{n}(t)$ as a background c-number. Note that the energies of the fermionic states are positive definite with vanishing vacuum energy in the adiabatic picture. The important point here is that we do not have any extra phase in (2.35), and we do not have any contribution from the fermionic part of the Hamiltonian for the evolution operator (2.36). This is consistent with the general relation

$$\begin{aligned}
&\det[\hbar\partial_\tau + \mu\sigma_3 - \mu_0 + U(\vec{n}(\tau))^\dagger\hbar\partial_\tau U(\vec{n}(\tau))] \\
&= \mathrm{Str}\{\exp[-(1/\hbar)\int_0^\beta H_{\text{fermion}}(\tau)d\tau]\} \\
&\sim \exp\{-\frac{1}{\hbar}\langle 0|H_{\text{fermion}}|0\rangle\beta\} = 1
\end{aligned} \tag{2.38}$$

for $\beta \to \infty$ with fixed large μ and μ_0 with $\mu_0 > \mu$ such that the vacuum with vanishing energy is isolated. When one defines the functional determinant with periodic boundary conditions, the determinant gives a supertrace. If one should have a Wess-Zumino term, the both-hand sides of this relation (2.38) would have an extra non-trivial phase factor relative to $\det[\hbar\partial_\tau + \mu\vec{n}(\tau)\cdot\vec{\sigma} - \mu_0]$. See eq.(3.25).

Instead of (2.38), one might prefer to consider (2.27) for $\beta \to$ large

$$\begin{aligned}
&\det[\hbar\partial_\tau + \mu\sigma_3 + U(\vec{n}(\tau))^\dagger\hbar\partial_\tau U(\vec{n}(\tau))] \\
&\sim \exp\{-\frac{1}{\hbar}\int_0^\beta d\tau\langle +|H_{\text{fermion}}|+\rangle\} \\
&= \exp\{\frac{\mu\beta}{\hbar} - i\oint\frac{(1+\cos\theta)}{2}d\varphi\}
\end{aligned} \tag{2.39}$$

with the fermionic Hamiltonian (2.17), for which one-fermion state with up-spin gives the energy lower than the vacuum [18]. It thus appears that one obtains the geometric term from the fermionic functional determinant in the leading term. This relation (2.39) is however ill-defined for $\beta \to \infty$ for which the geometric (adiabatic) phase is best defined [11]. Also, the vacuum and the state $|+-\rangle$ are degenerate in this case

$$\begin{aligned}
&\exp\{-\int_0^\beta d\tau \langle +-|H_{\rm fermion}|+-\rangle\} \\
&= \exp\{-\int_0^\beta d\tau \langle +|H_{\rm fermion}|+\rangle - \int_0^\beta d\tau \langle -|H_{\rm fermion}|-\rangle\} \\
&= \exp\{\frac{\mu\beta}{\hbar} - i\oint \frac{(1+\cos\theta)}{2} d\varphi - \frac{\mu\beta}{\hbar} - i\oint \frac{(1-\cos\theta)}{2} d\varphi\} \\
&= \exp\{-i\oint d\varphi\} = \exp\{-2\pi i\} = 1. \qquad (2.40)
\end{aligned}$$

It should be noted that the geometric terms appear in the sub-leading terms in (2.38). In this respect, it is immaterial if the geometric terms appear in the leading term or in the sub-leading terms by varying the parameter μ_0. The crucial property is that the Wess-Zumino term, if it should exist in the present model, should appear multiplying *all* the terms, not only the leading term but also the sub-leading terms in both of (2.38) and (2.39) when one starts, respectively, with the left-hand sides of (2.35) and (2.27). Obviously, no such a Wess-Zumino term in the present model. This may be compared to (3.26).

3 Field theoretical model; quantum anomaly

A unitary transformation and induced terms which are analogous to those discussed in the preceding section are realized by a field theoretical model defined by

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(x)[i\gamma^\mu(\partial_\mu - ieQA_\mu) - mU(\pi)]\psi(x) \\
&+ \frac{f_\pi^2}{16}{\rm Tr}\partial_\mu U(\pi)\partial^\mu U(\pi)^\dagger \qquad (3.1)
\end{aligned}$$

where

$$U(\pi) = e^{2i(1/f_\pi)\gamma_5\pi^a(x)T^a} \qquad (3.2)$$

and

$$\psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T^a = \frac{1}{2}\sigma^a. \tag{3.3}$$

In the present field theoretical model, we work in the Euclidean metric with $g_{\mu\nu} = (-1,-1,-1,-1)$. In this model $p(x)$ and $n(x)$ stand respectively for the idealized proton and neutron which are degenerate in mass, and $\pi^a(x)$ stand for the pion fields with σ^a standing for the Pauli matrix. $A_\mu(x)$ is the electromagnetic field. The above Lagrangian is invariant under the electromagnetic gauge transformation and also invariant under a global chiral $SU_L(2) \times SU_R(2)$ transformation which is weakly broken by the electromagnetic interaction. This chiral symmetry becomes explicit by writing the above Lagrangian as

$$\begin{aligned}\mathcal{L} &= \bar{\psi}_L(x) i\gamma^\mu(\partial_\mu - ieQA_\mu)\psi_L(x) + \bar{\psi}_R(x) i\gamma^\mu(\partial_\mu - ieQA_\mu)\psi_R(x) \\ &\quad - \bar{\psi}_L(x) m e^{2i(1/f_\pi)\pi^a(x)T^a}\psi_R(x) - \bar{\psi}_R(x) m e^{-2i(1/f_\pi)\pi^a(x)T^a}\psi_L(x) \\ &\quad + \frac{f_\pi^2}{16}\mathrm{Tr}\partial_\mu U(\pi)\partial^\mu U(\pi)^\dagger \end{aligned} \tag{3.4}$$

where

$$\psi_{L,R}(x) = (\frac{1 \mp \gamma_5}{2})\psi(x). \tag{3.5}$$

Under the global chiral transformation with global parameters $\chi^a T^a$,

$$\begin{aligned} \psi_L(x) &= e^{-i\chi^a T^a}\psi'_L(x), \\ \psi_R(x) &= e^{i\chi^a T^a}\psi'_R(x), \\ e^{2i(1/f_\pi)\pi^a(x)T^a} &= e^{-i\chi^a T^a} e^{2i(1/f_\pi)\pi^a(x)'T^a} e^{-i\chi^a T^a}, \end{aligned} \tag{3.6}$$

the Lagrangian is form invariant if one sets $e = 0$. If one imposes this global chiral symmetry, an additional naive mass term m_0 in (3.1) which is obtained by the replacement

$$mU(\pi) \to m_0 + mU(\pi) \tag{3.7}$$

is not allowed. This may be compared to (2.28).

We now perform a field-dependent unitary transformation

$$\begin{aligned} \psi(x) &= V(\pi)\psi'(x) = V_R(\pi)\psi'_R(x) + V_L(\pi)\psi'_L(x), \\ \bar{\psi}(x) &= \bar{\psi}'(x)V(\pi) = \bar{\psi}'_R(x)V_R(\pi)^\dagger + \bar{\psi}'_L(x)V_L(\pi)^\dagger \end{aligned} \tag{3.8}$$

with [5]

$$V(\pi) = e^{-i(1/f_\pi)\gamma_5\pi^a(x)T^a}. \tag{3.9}$$

[5] We have $V_R(\pi) = \exp\{-i\frac{1}{f_\pi}\pi^a T^a\}$ and $V_L(\pi) = \exp\{i\frac{1}{f_\pi}\pi^a T^a\}$ in the fixed chiral frame. If one defines the global chiral transformation law by $V_L(\pi) \to e^{-i\chi^a T^a}V_L(\pi)$ and $V_R(\pi) \to e^{i\chi^a T^a}V_R(\pi)$, the transformation law in (3.6) is realized if one understands that $\exp\{2i\frac{1}{f_\pi}\pi^a T^a\} = V_L(\pi)V_R(\pi)^\dagger$ and the fermion fields ψ' and $\bar{\psi}'$ in (3.8) are not transformed under the global chiral transformation.

One then naively obtains the result

$$\begin{aligned}
&\int \mathcal{D}U(\pi)\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{\int d^4x\mathcal{L}\} \\
&= \int \mathcal{D}U(\pi)\mathcal{D}\bar{\psi}'\mathcal{D}\psi' \exp\{\int d^4x\mathcal{L}'\} \\
&= \int \mathcal{D}U(\pi)\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{\int d^4x\mathcal{L}'\}
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\mathcal{L}' &= \bar{\psi}(x)[i\gamma^\mu(\partial_\mu - ieQA_\mu + V^\dagger(\pi)D_\mu V(\pi)) - m]\psi(x) \\
&\quad + \frac{f_\pi^2}{16}\mathrm{Tr}\partial_\mu U(\pi)\partial^\mu U(\pi)^\dagger
\end{aligned} \tag{3.11}$$

with

$$D_\mu V(\pi) = \partial_\mu V(\pi) - ie[QA_\mu, V(\pi)] \tag{3.12}$$

by assuming the invariance of the measure

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \mathcal{D}\bar{\psi}'\mathcal{D}\psi'. \tag{3.13}$$

We also used the fact that the naming of integral variables is arbitarary in the path integral (3.10).

Here we performed a naive manipulation by ignoring the possible Jacobian for the above change of integration variables (3.8). Nevertheless, we obtain the term in (3.11), which was called "Dyson term" in the old literature [25, 26, 27, 28]

$$\begin{aligned}
&\bar{\psi}(x)i\gamma^\mu(V^\dagger(\pi)D_\mu V(\pi))\psi(x) \\
&\sim (1/f_\pi)\bar{\psi}(x)\gamma^\mu\gamma_5(D_\mu\pi(x))\psi(x)
\end{aligned} \tag{3.14}$$

in the order linear in the variables $\pi(x)$ with $\pi(x) = \pi^a(x)T^a$ and

$$D_\mu\pi(x) = \partial_\mu\pi(x) - ie[QA_\mu, \pi(x)]. \tag{3.15}$$

The above naive manipulation suggests the equivalence of the derivative coupling in $\mathcal{L}'$ (3.11),

$$\mathcal{L}' \sim (1/f_\pi)\bar{\psi}(x)\gamma^\mu\gamma_5\partial_\mu\pi(x)\psi(x), \tag{3.16}$$

and the pseudoscalar coupling in the starting Lagrangian $\mathcal{L}$ (3.1),

$$\mathcal{L} \sim -2im(1/f_\pi)\bar{\psi}(x)\gamma_5\pi(x)\psi(x), \tag{3.17}$$

to the order linear in the pion fields.

The derivative coupling in $\mathcal{L}'$ (3.11), which appears sandwiched by fermion fields $\psi^\dagger$ and ψ, precisely corresponds to the geometric term in (2.16), though we here have a four-dimensional derivative instead of the simple time derivative in (2.16). Naively, the appearance of the derivative coupling is also regarded as a result of the failure of the decoupling theorem for $m \to$ large in the sense of Ref.[18], but the actual failure of the decoupling theorem is more involved as will be explained later. It is clear that the above operation is a naive one and the appearance of the above Dyson term has nothing to do with the quantum anomaly. It is well-known that the above two Lagrangians (3.1) and (3.11) give rise to quite different predictions for the decay amplitude $\pi^0 \to \gamma + \gamma$ in the *soft-pion* limit, which marked the genesis of the modern notion of quantum anomaly [29, 30]. This in particular implies that

$$\begin{aligned} &\mathrm{Det}[i\gamma^\mu(\partial_\mu - ieQA_\mu) - mU(\pi)] \\ &\neq \mathrm{Det}[i\gamma^\mu(\partial_\mu - ieQA_\mu + V^\dagger(\pi)D_\mu V(\pi)) - m] \end{aligned} \tag{3.18}$$

in contrast to (2.27) and (2.35).

Some of the essential and general properties of the quantum anomalies are:

1. The anomalies are not recognized by a naive manipulation of the classical Lagrangian or action (or by a naive canonical manipulation in operator formulation), which leads to the naive Nöther's theorem.
2. The quantum anomaly is related to the quantum breaking of classical symmetries (and the failure of the naive Nöther's theorem). For example, the Gauss law operator (or BRST charge) becomes time-dependent and thus it cannot be used to specify physical states in anomalous gauge theory [31].
3. The quantum anomalies are generally associated with an infinte number of degrees of freedom. The anomalies in the practical calculation are thus closely related to the regularization, though the anomalies by themselves are perfectly finite.
4. In the path integral formulation, the anomalies are recognized as nontrivial Jacobians for the change of path integral variables associated with classical symmetries.

None of these essential properties are shared with the derivation of geometric terms in Section 2. Rather, the geometric term there (2.16) corresponds to the naive Dyson term in (3.11), which is known to fail to account for the whole story of the above chiral transformation.

To incorporate the anomaly, one needs to evaluate the Jacobian carefully for the above chiral transformation (3.8) [32]. One may first rewrite the covariant derivative in (3.1) as

$$D_\mu = \partial_\mu - ieQA_\mu = \partial_\mu - ieYA_\mu - ieT^3 A_\mu \tag{3.19}$$

with

$$Y = \frac{1}{2}, \quad T^3 = \frac{1}{2}\sigma^3. \tag{3.20}$$

The Wess-Zumino term for the transformation (3.8) then has a well-known form [2, 3, 4]

$$\begin{aligned}
\mathcal{D}\bar{\psi}\mathcal{D}\psi &= J\mathcal{D}\bar{\psi}'\mathcal{D}\psi', \\
\ln J &= i\int d^4x \mathcal{L}_{\rm Wess-Zumino} \\
&= i\int d^4x \int_0^1 ds \frac{1}{f_\pi}\epsilon^{\mu\nu\alpha\beta}{\rm tr}\pi^a(x)T^a \frac{1}{16\pi^2} \\
&\quad \times\{\frac{e^2}{2}[U(s)^\dagger T^3 U(s) + U(s)T^3U(s)^\dagger]F_{\mu\nu}F_{\alpha\beta}] \\
&\quad\quad +4ie[F_{\mu\nu}a_\alpha a_\beta]\}
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
&F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \\
&a_\alpha = \frac{i}{2}[U(s)^\dagger D_\alpha U(s) - U(s)(D_\alpha U(s))^\dagger]
\end{aligned} \tag{3.22}$$

with

$$\begin{aligned}
&U(s) \equiv e^{-is(1/f_\pi)\pi^a(x)T^a}, \\
&D_\alpha U(s) = \partial_\alpha U(s) - ie[A_\alpha T^3, U(s)].
\end{aligned} \tag{3.23}$$

This is obtained by an integral of the Jacobian for the repeated applications of the infinitesimal transformation

$$\begin{aligned}
\psi(x) &= e^{-ids(1/f_\pi)\pi^a(x)T^a\gamma_5}\psi'(x), \\
\bar{\psi}(x) &= \bar{\psi}'(x)e^{-ids(1/f_\pi)\pi^a(x)T^a\gamma_5},
\end{aligned} \tag{3.24}$$

and tr stands for the trace over the 2×2 matrices with ${\rm tr}T^aT^b = \frac{1}{2}\delta_{ab}$.

In terms of the functional determinant we have [6]

$$\begin{aligned}
&{\rm Det}[i\gamma^\mu(\partial_\mu - ieQA_\mu) - mU(\pi)] \\
&= {\rm Det}[i\gamma^\mu(\partial_\mu - ieQA_\mu + V^\dagger(\pi)D_\mu V(\pi)) - m] \\
&\quad \times \exp\{i\int d^4x\mathcal{L}_{\rm Wess-Zumino}\}
\end{aligned} \tag{3.25}$$

[6] In the present chiral $SU(2)$ symmetry, which is anomaly free by itself, no Wess-Zumino term arises for $A_\mu = 0$. For $SU(3)$, for example, one obtains a non-trivial Jacobian or the Wess-Zumino term even with $A_\mu = 0$, and such a term is shown to exhibit a topological property [2, 3].

which may be compared to (2.27). For $T \to$ large (in the present Euclidean theory), we have

$$\begin{aligned}
&\mathrm{Det}[i\gamma^\mu(\partial_\mu - ieQA_\mu + V^\dagger(\pi)D_\mu V(\pi)) - m] \\
&\quad \times \exp\{i\int d^4x \mathcal{L}_{\mathrm{Wess-Zumino}}\} \\
&\sim \exp\{-E_{\mathrm{vac}}T\}\exp\{i\int d^4x \mathcal{L}_{\mathrm{Wess-Zumino}}\} \\
&= \exp\{i\int d^4x \mathcal{L}_{\mathrm{Wess-Zumino}}\} \qquad (3.26)
\end{aligned}$$

for a fixed large m and slowly varying $\pi(x)$ with periodic boundary conditions, for which we have a mass gap $\sim m$ and thus the fermionic vacuum with vanishing energy is isolated. This relation may be compared to (2.38).

In the order linear in the pion fields, we have the Jacobian

$$\begin{aligned}
\ln J &= i\int d^4x \frac{1}{f_\pi}(\mathrm{tr}T^aT^3)\pi^a(x)\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \\
&= i\int d^4x \frac{1}{f_\pi}\pi^0(x)\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \qquad (3.27)
\end{aligned}$$

It is well known that this Wess-Zumino term (3.27) when added to the Lagrangian $\mathcal{L}'$ in (3.11)

$$\begin{aligned}
&\int d^4x[\mathcal{L}' + \mathcal{L}_{\mathrm{Wess-Zumino}}] \\
\sim & -\int d^4x(1/f_\pi)\pi^0(x)\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5T^3\psi(x)] \\
& + \int d^4x\frac{i}{f_\pi}\pi^0(x)\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \qquad (3.28)
\end{aligned}$$

correctly describes the decay $\pi^0 \to \gamma + \gamma$ in the soft-pion limit [29, 30] in agreement with the result on the basis of (3.1). In the operator notation, this equivalence is expressed as the relation

$$\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5T^3\psi(x)] = 2im[\bar{\psi}(x)\gamma_5T^3\psi(x)] + i\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \qquad (3.29)$$

which expresses the failure of the naive Nöther's theorem for the exact global chiral symmetry generated by γ_5T^3.

It is instructive to see in detail how this equivalence in the decay $\pi^0 \to \gamma + \gamma$ is achieved. We consider two distinct cases:
(i)$m \neq 0$
In this case, the operator $[\bar{\psi}(x)\gamma^\mu\gamma_5T^3\psi(x)]$ in (3.28) is free of infrared singularity in the soft-pion limit, namely, for the four-momentum of the pion $p_\mu \sim 0$. Thus the first term in (3.28) vanishes in the soft-pion limit

$$\lim_{p_\mu\to 0}\int d^4x e^{ip_\mu x^\mu}\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5T^3\psi(x)] = 0 \qquad (3.30)$$

and the second anomaly term gives the same result as the pseudo-scalar coupling in (3.17).

(ii) $m = 0$

This case is singular from the point of view of spontaneously broken chiral symmetry. Nevertheless, in this case, which corresponds to the case $\mu = 0$ in (2.2), the pseudo-scalar coupling in (3.17) vanishes. On the other hand, the current $[\bar{\psi}(x)\gamma^{\mu}\gamma_5 T^3\psi(x)]$ becomes singular in the soft-pion limit but still one can use the operator relation (3.29) with $m = 0$ in (3.28) and the two terms in (3.28) cancel each other, to be consistent with the vanishing pseudo-scalar coupling in (3.17).

We emphasize that the derivation of the chiral anomaly does not depend on the relative magnitude of m, which is analogous to μ in (2.2), and the frequency of the external variables such as the gauge field, but rather depends on the relative magnitude of the cut-off mass M, such as the Pauli-Villars regulator mass which can be chosen to be arbitrarily large, and the frequency of the external variables. This is quite different from the case of the geometric phase where the parameter μ, which corresponds to m, directly enters the criterion of the adiabatic approximation in (2.20) where $1/T$ corresponds to the frequency of the external variables. Because of this difference, the *failure* of the decoupling theorem in the chiral anomaly is stated precisely as follows: The derivative coupling term in (3.28), which corresponds to the geometric term in (2.16), vanishes for $m \to \infty$ [1], and the anomaly term balances the pseudo-scalar term in (3.17) which does not vanish in the limit.

The discovery of the chiral anomaly is based on the recognition that the naive Dyson's relation, namely the naive equivalence between (3.1) and (3.11), inevitably fails in gauge field theory, and one needs to include an extra Jacobian (or Wess-Zumino term).

4 Discussion

We have explained the basic differences between the geometric phase and the quantum anomaly by analyzing the concrete model due to Stone and a corresponding model in field theory which contains a true quantum anomaly. The only similarity between the geometric phase in the adiabatic approximation and the Wess-Zumino term is that both of them exhibit topological properties under certain limiting conditions.

In contrast, the differences are more profound and fundamental. Firstly, the geometric phase arises from the *naive* rearrangement of terms inside the fermionic operator sandwiched by $\psi^{\dagger}$ and ψ, whereas the Wess-Zumino term associated with quantum anomaly arises from the Jacobian, namely, a completely new additional part to the Lagrangian. Secondly, the geometric

phase is recognized only when one throws away a part of the original Hamiltonian in the adiabatic approximation, whereas the quantum anomaly is exact without any approximation. The Born-Oppenheimer approximation in the geometric phase means a neglect of a part of the Hamiltonian, whereas the Born-Oppenheimer approximation in the quantum anomaly, if any [24], is actually not an approximation; this is obvious in the path integral formulation of quantum anomalies where the Born-Oppenheimer approximation simply means a specific order of path integration, namely, one first integrates over fermions with fixed bosonic (such as gauge or Nambu-Goldstone) variables and then one integrates over bosonic variables later. The path integral over the fermionic variables, which are quadratic, can be performed exactly and the Wess-Zumino term is induced by this fermionic path integral; in this sense no approximation is invloved in the analysis of the chiral anomaly, though the path integral of bosonic variables in the non-linear effective chiral model in Section 3 is not renormalizable. Because of this difference, the topological property of the geometric phase is inevitably trivial in the practical Born-Oppenheimer approximation for any finite time interval T if one deals with the exact Hamiltonian, whereas the topology in the quantum anomaly, which is basically a short distance phenomenon in four-dimesional space-time, is exact once its existence is established since no approximation is involved.

One may still wonder, if our assertion is valid, what then happens with the analysis by Nelson and Alvarez-Gaume [24] where a precise analogy between the quantum anomaly in the Hamiltonian interpretation and the geometric phase is forcefully argued. We believe that all what are said about the chiral anomaly there [24] are correct. We also believe that they can perform all of their analyses of the chiral anomaly without referring to the geometric phase in quantum mechanics. The pair production picture in [24] is based on the fact that one can arrive at the level crossing point with vanishing energy at a fixed well-defined time t_0. This means the *failure* of the naive adiabatic picture as is emphasized in [24]. On the other hand, the validity of the topological property of the geometric phase is based on the condition that we never approach the level crossing point for any finite t, namely, on the strict validity of the adiabatic picture. The topological property of the geometric phase cannot be used in the context of the analysis in [24]. If one arrives at the level crossing at finte $t = t_0$, for example, one can suitably redefine the time variable and smoothly deform the background variable such that the period T of the background variable is finite. This is generally the case in the mathematical analysis of the index theorem [36] which is based on the compact Euclidean space-time such as S^4. For a finite time interval T, the topological property of the geometric phase, such as the topological proof [12] of the Longuet-Higgins phase change rule, fails as we have shown

elsewhere [21] and also in Section 2 of the present paper. The topological property of the geometric phase crucially depends on the very precise definition of the adiabatic approximation; the movement of the external parameter must be infinitely slow, *i.e.*, not only the period $T \to \infty$ but also the variation of the background variable at each moment is negligible [11].

As is clear in the analysis in [24], the quantum anomaly influences all the states of the Fock space equally, whereas the geometric phase appears only in the specific states of the Fock space and does not influence the vacuum state. The form of the geometric phase is also state-dependent. See eq.(2.38). Also, the geometric phase is indepedent of the parameter μ_0 as in (2.33), whereas the analysis of quantum anomaly in [24] crucially depends on the crossing of vanishing eigenvalues in chiral gauge theory which corresponds to the specific choice $\mu_0 = 0$ in the context of the model in Section 2. The general level crossing problem in the context of geometric phases is regarded to be related to the vector-like transformation as is seen in (2.33) and (2.35). See also (A.9). In contrast the chiral symmetry, not the vector-like symmetry, is crucial in the analysis of the anomaly. The level crossing by itself has no connection to the anomaly.

We also note that all the properties of the chiral anomaly are understood in terms of the Green's functions instead of going to the S-matrix. The Green's functions are the statements about the local properties of field theory unlike the S-matrix which involves a subtle limit of the infinite time interval in field theory. The global $SU(2)$ anomaly by Witten [33] may appear to depend on the infinite time interval to some extent, but we note that the global $SU(2)$ anomaly is also known to be described by the Wess-Zumino term related to the group $SU(3)$, which is defined in the framework of Green's functions, in a suitable fomulation of the problem [34, 35]. The quantum anomaly, as we understand it in gauge field theory, is a precise statement and as such it should not depend on the technical details of the adiabatic approximation, unlike the case of the geometric phase associated with level crossing in quantum mechanics.

There are well-known odd-dimensional cousins of chiral anomalies, namely, the Chern-Simons terms which exhibit topological properties. The Chern-Simons terms induced by fermions, which are sometimes called parity anomaly, or added by hand are closely related to the chiral anomaly not only by the descent formula [2, 3] or dimensional reduction but also in the explicit Feynman diagramatic calculations. If one provides a precise definition of the geometric (or Berry) phase in general field theoretical contexts, possibly asking some association with level crossing and adiabaticity as minimal requirements, it would be possible to analyze the relation between the Berry phase and the odd-dimensional cousins of chiral anomaly.

As an explicit example of the geometric phase in realistic condensed mat-

ter physics, we mention the recent works on anomalous Hall effect [37]. In those works, readers will find that all the basic ingredients of the geometric (or Berry) phase, such as level crossing, adiabaticity and approximate topology, are contained. This class of models are included in the general model in Appendix of the present paper, and thus our analysis in the present paper is applicable to them.

Finally, we note that the Aharonov-Bohm phase is topologically exact even for a finite time interval T unlike the geometric phase. The Aharonov-Bohm effect contains an extra dynamical freedom, namely, the electromagnetic potential which is *time-independent*, and the space for the Aharonov-Bohm effect is not simply connected. None of these crucial features are shared with the geometric phase, though certain feature of the Aharonov-Bohm effect is known to be shared with the geometric phase [10]. We think that a clear distinction between the Aharonov-Bohm phase and the geometric phase is also important, since the notion of winding number is defined for the Aharonov-Bohm phase whereas no notion of winding number in the geometric phase for any finite time interval T as the topology is trivial.

5 Conclusion

The model in Ref. [18], which is essentially identical to the simplest example discussed by Berry in his original paper [10], shows that the Berry phase associated with level crossing gives the topological phase for certain states in the Fock space in the precise adiabatic limit. The phase factor has the same form as the anomaly-induced Wess-Zumino term appearing in certain field theoretical models. The key concepts involved in the model, namely, the level crossing, topology and adiabatic approximation also appear in the Hamiltonian analysis of chiral anomalies by Nelson and Alvarez-Gaume [24]. This fact led to an expectation that the very basic mechanism of chiral anomalies, which have been established by the efforts of various authors, notably by Bell and Jackiw [29] and Adler [30], may be identified with the basic mechanism of the adiabatic Berry phase related to level crossing in the simple Schrödinger problem. What we have shown in the present paper is that this expectation is not realized, and the similarity between the two is superficial. We have first explained the difference between the two on the basis of general characteristics of chiral anomaly, such as the failure of the naive manipulation and the failure of the decoupling theorem, by using two explicit examples in Sections 2 and 3. Our conclusion is valid for a more general class of level crossing problems summarized in Appendix. We then explained the difference between the two from the point of view of level crossing and topology. The difference between the chiral anomaly and the Berry phase is

simply stated as follows: The topology of gauge fields leads to level crossing in the fermionic sector in the case of chiral anomaly and the *failure* of the adiabatic approximation is essential in the analysis, whereas the (potential) level crossing in the matter sector leads to the topology of the Berry phase only when the very precise adiabatic approximation holds. These two cannot be compatible with each other.

In the early literature on the geometric phase, the similarity between the geometric phase and the quantum anomaly, though rather superficial one, was emphasized [38]. That analogy was useful at the initial developing stage of the subject. But in view of the wide use of the terminology "geometric phase" in various fields in physics today [39], it is our opinion that a more precise distinction of various loosely related phenomena is also desirable. To be precise, what we are suggesting is to call chiral anomaly as chiral anomaly, Wess-Zumino term as Wess-Zumino term, Chern-Simons term as Chern-Simons term, and Aharovov-Bohm phase as Aharonov-Bohm phase, etc., since those terminologies convey very clear messages and well-defined physical contents which the majority in physics community can readily recognize. Even in this sharp definition of terminology, one can still clearly identify the geometric (or Berry) phase and its physical characteristics, which cannot be described by other notions, as the concrete physical example in Ref. [37] suggests.

We believe that a sharp definition of the scientific term "geometric phase", probably by asking some association with level crossing and adiabaticity as minimal requirements, is also important for those experts working on the geometric phase itself, since then the wider audience can easily identify the phenomena, which are intrinsic to the geometric phase and cannot be described by other notions, and consequently they will appreciate more the usefulness of the geometric phase.

A General Level Crossing Problem

The general geometric phase associated with any level crossing in the second quantized formulation exhibits the same topological properties as the specific example in Section 2; approximate monopole-like behavior in the adiabatic approximation but actually topologically trivial in the infinitesimal neighborhood of level crossing for any finite time interval T. This property may be relevant to the analysis in Ref.[19], where the geometric phase is used as an *analogue* of the Wess-Zumino term, and we sketch the analysis of the general level crossing [21, 22] in this appendix:

We start with the generic hermitian Hamiltonian

$$\hat{H} = \hat{H}(\hat{\vec{p}}, \hat{\vec{x}}, X(t)) \tag{A.1}$$

for a single particle theory in the background variable $X(t) = (X_1(t), X_2(t), ...)$. The path integral for this theory for the time interval $0 \le t \le T$ in the second quantized formulation is given by

$$\begin{aligned} Z &= \int \mathcal{D}\psi^\star \mathcal{D}\psi \exp\{\frac{i}{\hbar}\int_0^T dt d^3x [\psi^\star(t,\vec{x}) i\hbar \frac{\partial}{\partial t}\psi(t,\vec{x}) \\ &\quad -\psi^\star(t,\vec{x})\hat{H}(\frac{\hbar}{i}\frac{\partial}{\partial \vec{x}}, \vec{x}, X(t))\psi(t,\vec{x})]\}. \end{aligned} \tag{A.2}$$

We then define a complete set of eigenfunctions

$$\begin{aligned} &\hat{H}(\frac{\hbar}{i}\frac{\partial}{\partial \vec{x}}, \vec{x}, X(0))u_n(\vec{x}, X(0)) = \lambda_n u_n(\vec{x}, X(0)), \\ &\int d^3x u_n^\star(\vec{x}, X(0)) u_m(\vec{x}, X(0)) = \delta_{nm}, \end{aligned} \tag{A.3}$$

and expand

$$\psi(t,\vec{x}) = \sum_n a_n(t) u_n(\vec{x}, X(0)). \tag{A.4}$$

We then have

$$\mathcal{D}\psi^\star \mathcal{D}\psi = \prod_n \mathcal{D}a_n^\star \mathcal{D}a_n \tag{A.5}$$

and the path integral is written as

$$\begin{aligned} Z &= \int \prod_n \mathcal{D}a_n^\star \mathcal{D}a_n \exp\{\frac{i}{\hbar}\int_0^T dt [\sum_n a_n^\star(t) i\hbar \frac{\partial}{\partial t} a_n(t) \\ &\quad -\sum_{n,m} a_n^\star(t) E_{nm}(X(t)) a_m(t)]\} \end{aligned} \tag{A.6}$$

where

$$E_{nm}(X(t)) = \int d^3x u_n^\star(\vec{x}, X(0)) \hat{H}(\frac{\hbar}{i}\frac{\partial}{\partial \vec{x}}, \vec{x}, X(t)) u_m(\vec{x}, X(0)). \tag{A.7}$$

We next perform a unitary transformation

$$a_n = \sum_m U(X(t))_{nm} b_m \tag{A.8}$$

where

$$U(X(t))_{nm} = \int d^3x u_n^\star(\vec{x}, X(0)) v_m(\vec{x}, X(t)) \tag{A.9}$$

with the instantaneous eigenfunctions of the Hamiltonian

$$\begin{aligned} &\hat{H}(\frac{\hbar}{i}\frac{\partial}{\partial \vec{x}}, \vec{x}, X(t)) v_n(\vec{x}, X(t)) = \mathcal{E}_n(X(t)) v_n(\vec{x}, X(t)), \\ &\int d^3x v_n^\star(\vec{x}, X(t)) v_m(\vec{x}, X(t)) = \delta_{n,m}. \end{aligned} \tag{A.10}$$

We can thus re-write the path integral as

$$Z = \int \prod_n \mathcal{D}b_n^\star \mathcal{D}b_n \exp\{\frac{i}{\hbar}\int_0^T dt[\sum_n b_n^\star(t) i\hbar\frac{\partial}{\partial t}b_n(t)$$
$$+\sum_{n,m} b_n^\star(t)\langle n|i\hbar\frac{\partial}{\partial t}|m\rangle b_m(t) - \sum_n b_n^\star(t)\mathcal{E}_n(X(t))b_n(t)]\} \quad \text{(A.11)}$$

where the second term in the action, which is defined by

$$\int d^3x v_n^\star(\vec{x}, X(t)) i\hbar\frac{\partial}{\partial t}v_m(\vec{x}, X(t)) \equiv \langle n|i\hbar\frac{\partial}{\partial t}|m\rangle, \quad \text{(A.12)}$$

stands for the geometric term. We take the time T as a period of the variable $X(t)$. The adiabatic process means that T is much larger than the typical time scale $\hbar/\Delta\mathcal{E}_n(X(t))$. The result (A.11) is also directly obtained by the expansion

$$\psi(t,\vec{x}) = \sum_n b_n(t)v_n(\vec{x}, X(t)). \quad \text{(A.13)}$$

In the operator formulation, we thus obtain the effective Hamiltonian (depending on Bose or Fermi statistics)

$$\begin{aligned}\hat{H}_{eff}(t) &= \sum_n \hat{b}_n^\dagger(t)\mathcal{E}_n(X(t))\hat{b}_n(t) \\ &\quad - \sum_{n,m} \hat{b}_n^\dagger(t)\langle n|i\hbar\frac{\partial}{\partial t}|m\rangle \hat{b}_m(t)\end{aligned} \quad \text{(A.14)}$$

with $[\hat{b}_n(t), \hat{b}_m^\dagger(t)]_\mp = \delta_{n,m}$. All the information about geometric phases is included in the effective Hamiltonian and thus geometric phases are purely *dynamical*.

When one defines the Schrödinger picture $\hat{\mathcal{H}}_{eff}(t)$ by replacing all $\hat{b}_n(t)$ by $\hat{b}_n(0)$ in $\hat{H}_{eff}(t)$, the second quantization formula for the evolution operator gives rise to [21, 22]

$$\begin{aligned}&\langle m|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{\mathcal{H}}_{eff}(t)dt\}|n\rangle \\ &= \langle m(t)|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{H}(\hat{\vec{p}}, \hat{\vec{x}}, X(t))dt\}|n(0)\rangle\end{aligned} \quad \text{(A.15)}$$

where $T^\star$ stands for the time ordering operation, and the state vectors in the second quantization on the left-hand side are defined by $|n\rangle = \hat{b}_n^\dagger(0)|0\rangle$, and the state vectors on the right-hand side stand for the first quantized states defined by $\langle\vec{x}|n(t)\rangle = v_n(\vec{x}, (X(t))$. Both-hand sides of the above equality (A.15) are exact, but the difference is that the geometric term, both

of diagonal and off-diagonal, is explicit in the second quantized formulation on the left-hand side.

The probability amplitude which satisfies Schrödinger equation is given by

$$\psi_n(\vec{x},t;X(t)) = \langle 0|\hat{\psi}(t,\vec{x})\hat{b}_n^\dagger(0)|0\rangle \tag{A.16}$$

since $i\hbar\partial_t\hat{\psi} = \hat{H}\hat{\psi}$ in the present problem. In the adiabatic approximation, where we assume the dominance of diagonal elements, we have (see also [15])

$$\begin{aligned}
&\psi_n(\vec{x},t;X(t)) \\
&= \sum_m v_m(\vec{x};X(t))\langle m(t)|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{H}(\hat{\vec{p}},\hat{\vec{x}},X(t))dt\}|n(0)\rangle \\
&= \sum_m v_m(\vec{x};X(t))\langle m|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{\mathcal{H}}_{eff}(t)dt\}|n\rangle \\
&\simeq v_n(\vec{x};X(t)) \exp\{-\frac{i}{\hbar}\int_0^t [\mathcal{E}_n(X(t)) - \langle n|i\hbar\frac{\partial}{\partial t}|n\rangle]dt\}.
\end{aligned} \tag{A.17}$$

by noting (A.15).

The path integral formula (A.11) is based on the expansion (A.13) and the starting path integral (A.2) depends only on the field variable $\psi(t,\vec{x})$, not on $\{b_n(t)\}$ and $\{v_n(\vec{x},X(t))\}$ separately. This fact shows that our formulation contains an exact hidden local gauge symmetry

$$\begin{aligned}
&v_n(\vec{x},X(t)) \rightarrow v_n'(t;\vec{x},X(t)) = e^{i\alpha_n(t)}v_n(\vec{x},X(t)), \\
&b_n(t) \rightarrow b_n'(t) = e^{-i\alpha_n(t)}b_n(t), \qquad n = 1,2,3,...,
\end{aligned} \tag{A.18}$$

where the gauge parameter $\alpha_n(t)$ is a general function of t. One can confirm that the action and the path integral measure in (A.11) are both invariant under this gauge transformation. This local symmetry is exact as long as the basis set is not singular, and thus it is particularly useful in the general adiabatic approximation defined by the condition that the basis set (A.10) is well-defined [7]. The specific basis set (A.10) becomes singular on top of level crossing. Of course, one may consider a new hidden local gauge symmetry when one defines a new regular basis set in the neighborhood of the singularity, and the freedom in the phase choice of the new basis set persists.

The above hidden local gauge symmetry (A.18) is an exact symmetry of quantum theory, and thus physical observables in the adiabatic approximation should respect this symmetry. Also, by using this local gauge freedom,

[7] This symmetry is a statement that the choice of the coordinates in the functional space is arbitrary in field theory. This symmetry by itself does not imply any conservation law. If one neglects the off-diagonal parts of the geometric term, the theory becomes invariant under $b_n(t) \rightarrow b_n' = e^{-i\alpha_n}b_n(t)$ for a constant α_n with fixed $v_n(\vec{x},X(t))$, and then the symmetry implies a (rather trivial) conservation law, namely, no level crossing.

one can choose the phase convention of the basis set $\{v_n(t,\vec{x},X(t))\}$ at one's will such that the analysis of geometric phases becomes simplest.

Our next observation is that $\psi_n(\vec{x},t;X(t))$ transforms under the hidden local gauge symmetry (A.18) as

$$\psi_n'(\vec{x},t;X(t)) = e^{i\alpha_n(0)}\psi_n(\vec{x},t;X(t)) \tag{A.19}$$

independently of the value of t. This transformation is derived by using the exact representation (A.16). This transformation is explicitly checked for the adiabatic approximation (A.17) also.

Thus the product

$$\psi_n(\vec{x},0;X(0))^\star\psi_n(\vec{x},T;X(T)) \tag{A.20}$$

defines a manifestly gauge invariant quantity, namely, it is independent of the choice of the phase convention of the complete basis set $\{v_n(t,\vec{x},X(t))\}$. One may employ this (rather strong) gauge invariance condition as the basis of the analysis of geometric phases, which is shown to replace the notions of parallel transport and holonomy [22]. Our hidden local gauge symmetry is a symmetry of quantum theory and that the Schrödinger amplitude $\psi_n(\vec{x},t;X(t))$ stays in the same ray under an arbitrary hidden local gauge transformation of the basis set as is shown in (A.19).

For the adiabatic formula (A.17), the gauge invariant quantity (A.20) is given by

$$\begin{aligned}
&\psi_n(\vec{x},0;X(0))^\star\psi_n(\vec{x},T;X(T))\\
&= v_n(0,\vec{x};X(0))^\star v_n(T,\vec{x};X(T))\\
&\times\exp\{-\frac{i}{\hbar}\int_0^T[\mathcal{E}_n(X(t)) - \langle n|i\hbar\frac{\partial}{\partial t}|n\rangle]dt\}
\end{aligned} \tag{A.21}$$

where we used the notation $v_n(t,\vec{x};X(t))$ to emphasize the use of arbitrary gauge in this expression. We then observe that by choosing the gauge such that $v_n(T,\vec{x};X(T)) = v_n(0,\vec{x};X(0))$ the prefactor $v_n(0,\vec{x};X(0))^\star v_n(T,\vec{x};X(T))$ becomes real and positive. Note that we are assuming the cyclic evolution of the external parameter, $X(T)=X(0)$. Then the factor

$$\exp\{-\frac{i}{\hbar}\int_0^T[\mathcal{E}_n(X(t)) - \langle n|i\hbar\frac{\partial}{\partial t}|n\rangle]dt\} \tag{A.22}$$

extracts all the information about the phase in (A.21) and defines a physical quantity. After this gauge fixing, the above quantity (A.22) is still invariant under residual gauge transformations satisfying the periodic boundary condition $\alpha_n(0)=\alpha_n(T)$, in particular, for a class of gauge transformations defined by $\alpha_n(X(t))$. Note that our gauge transformation in (A.18), which is defined by an arbitrary function $\alpha_n(t)$, is much more general.

In the analysis of the behavior in the infinitesimal neighborhood of a specific level crossing, one may truncate the above general model to a two-level model containing the two levels at issue, and the present formulation (A.14) is essentially reduced to the model (2.17) or (2.18); one then finds the same approximate topological property for any finite T as in the model (2.17). This is explained in detail in Ref. [21].

Based on the above general analysis, the essence of geometric phase may be summarized as follows: One obtains an interesting universal view such as in (A.22) about various level crossing problems by making an *approximation* (adiabatic approximation), which is not clearly seen in the exact treatment on the right-hand side of (A.15).

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